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Inconsistency of Nonlinear Second-Order Equations of Motion of Classical Charges

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Problems of second-order equations of motion of elementary classical charges are discussed. Inconsistency of energy-momentum balance is pointed out. In particular, it is shown, probably for the first time, that Eliezer's equation does not conserve energy. The results favor the third-order equation of Lorentz–Dirac.

1. INTRODUCTION

The equation of motion of a classical point charge is a debated subject. The existence of several, mutually contradictory, equations of motion provide an indication of this situation. Some differential equations of this kind take the second-order Newtonian form and depend *nonlinearly* on external fields (Eliezer, 1948; Mo and Papas, 1971; Bonnor, 1974; Herrera, 1977). As a matter of fact, specific proofs showing that all but the first of these equations are unphysical have already been published by Huschilt and Baylis (1974) and Comay (1987). However, not all these equations have been refuted. Moreover, other equations belonging to this category may be published in the future. The present work points out an inherent problem which is common to all equations of this kind. This problem is unsettled by the above-mentioned equations and is the underlying reason for their unphysical properties.

This work uses units where the speed of light $c = 1$. Greek indices range from 0 to 3 and Latin ones range from 1 to 3. The symbol $_{,\nu}$ denotes the partial differentiation with respect to x^{ν}. The metric is diagonal and its entries are $(1, -1, -1, -1)$. τ denotes the invariant time.

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A fundamental property of relativity is the finite velocity of energy-momentum. As a result, energy-momentum move continuously and their density and flux must be defined at points of the medium encompassing interacting charges. In the classical electrodynamics of continuously charged matter, this requirement is accomplished by the construction of an energy-momentum tensor of fields (Landau and Lifshitz, 1975, p. 81)

$$T^{\mu\nu}_{(f)} = \frac{1}{4\pi}(F^{\mu\alpha}F^{\beta\nu}g_{\alpha\beta} + \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}) \tag{1}$$

where $F^{\mu\nu}$ is the tensor of fields

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2}$$

The entries $T^{\nu 0}$ are energy-momentum densities and $T^{\nu i}$ are energy-momentum currents.

An electromagnetic energy-momentum tensor has to satisfy the requirement

$$T^{\mu\nu}_{(f),\nu} = -F^{\mu\nu}J_\nu \tag{3}$$

This relation shows that the theory conserves energy-momentum at vacuum points where $J^\mu = 0$. Using (3) together with the Lorentz force

$$\frac{dP^\mu}{d\tau} = F^{\mu\nu}J_\nu \tag{4}$$

one verifies energy-momentum conservation at points where charge does not vanish. Thus, classical electrodynamics of continuously charged matter conserves energy-momentum.

2. THE PROBLEM OF POINT CHARGES

The introduction of elementary point charges alters the situation. The elementary nature of such a particle means that one cannot consider it as being made of distinct constituents (see, e.g., Rohrlich, 1965, pp. 127–129; Landau and Lifshitz, 1975, pp. 43–44). In the case of continuously charged matter, effects of q^2 can be ignored because one can always look at an infinitesimal amount of charge. The interaction of this charge with an external field is linear in q, justifying the elimination of terms containing higher powers of q. This procedure is inapplicable to an elementary classical point charge q. For such a particle, effects of q^2 prove that it cannot satisfy the Lorentz force (4). Indeed, take, for example, a light negative charge attracted by a massive positive charge which is held fixed at the origin. The velocity of the negative charge is perpendicular to its radius vector and all quantities are calibrated so that the attraction of the negative charge toward the origin provides the centripetal force required for keeping a uniform circular motion. Therefore, if an elementary point charge interacts only with external forces and if the Lorentz force (4) determines the charge's equation of motion, then one can build a closed system that emits radiation and whose state does not vary in time. Obviously, this process does not conserve energy and is physically unacceptable.

An analysis of the motion of a classical point charge enabled Lorentz and Abraham to derive the Lorentz-Dirac (LD) equation (Rohrlich, 1965, pp. 11-18; Klepikov, 1985). This equation was derived later by other authors on the basis of various physical arguments (Dirac, 1938; Infeld and Wallace, 1940; Wheeler and Feynman, 1945; Landau and Lifshitz, 1975; Teitlboim, 1970; Barut, 1974). The LD equation can be put in the following form:

$$\tfrac{2}{3}q^2\frac{da^\mu}{d\tau} = ma^\mu - qF^{\mu\nu}_{\text{ext}}v_\nu - \tfrac{2}{3}q^2(a^\alpha a_\alpha)v^\mu \tag{5}$$

where m and q denote the mass and charge of the particle, respectively and $F^{\mu\nu}_{\text{ext}}$ is the tensor of the external fields, i.e., fields not associated with the particle's charge q. The LD equation is a third-order differential equation, which is inconsistent with the second-order Newtonian form. This point has motivated several authors to suggest alternative equations of motion of charges. Some of the suggested equations take the second-order Newtonian form and are characterized by a nonlinear dependence on external fields,

$$ma^\mu = qF^{\mu\nu}_{\text{ext}}v_\nu + \frac{2q^3}{3m}\left[\frac{d(F^{\mu\nu}_{\text{ext}}v_\nu)}{d\tau} + (F^{\alpha\beta}_{\text{ext}}a_\alpha v_\beta)v^\mu\right] \tag{6}$$

$$ma^\mu = qF^{\mu\nu}_{\text{ext}}v_\nu + \frac{2q^3}{3m}[F^{\mu\nu}_{\text{ext}}a_\nu + (F^{\alpha\beta}_{\text{ext}}a_\alpha v_\beta)v^\mu] \tag{7}$$

$$ma^\mu = qF^{\mu\nu}_{\text{ext}}v_\nu + \frac{2q^4}{3m^2}[F^{\mu\alpha}_{\text{ext}}g_{\alpha\lambda}F^{\lambda\beta}_{\text{ext}}v_\beta + (F^{\alpha\beta}_{\text{ext}}v_\beta g_{\alpha\lambda}F^{\lambda\nu}_{\text{ext}}v_\nu)v^\mu] \tag{8}$$

$$\frac{d(mv^\mu)}{d\tau} = qF^{\mu\nu}_{\text{ext}}v_\nu + \tfrac{2}{3}q^2(a^\alpha a_\alpha)v^\mu \tag{9}$$

Equations (6)-(9) have been suggested respectively by Eliezer (1948), Mo and Papas (1971), Herrera (1977), and Bonnor (1974).

In (8) one sees explicitly that the 4-acceleration depends nonlinearly on fields. The other equations, (6), (7), and (9), are implicit expressions where a^μ appears on both sides. It is easy to see that, casting each of these equations into an explicit expression of a^μ in terms of $F^{\mu\nu}_{\text{ext}}$ and v^λ, one obtains a nonlinear dependence of a^μ on external fields. Problems of the energy balance of such nonlinear equations are discussed in this work.

3. ENERGY-MOMENTUM AND THE LORENTZ–DIRAC EQUATION

Consider the dimensions of the fields' energy-momentum tensor $T^{\mu\nu}_{(f)}$. This discussion is carried out in units where $\hbar = c = 1$. In these units charge and velocity are dimensionless, energy is measured in units of L^{-1}, and electromagnetic fields are measured in units of L^{-2}. As is well known, $T^{00}_{(f)}$ is the fields' energy density (Landau and Lifshitz, 1975, p. 78). Therefore, it is measured in units of L^{-4}. Since $T^{\mu\nu}_{(f)}$ is a function of fields alone, it must be a homogeneous quadratic function of $F^{\mu\nu}$. This requirement is satisfied by (1). The energy-momentum gained by fields is the 4-divergence $T^{\mu\nu}_{(f),\nu}$. Using the Maxwell equation $F^{\mu\nu}{}_{,\nu} = -4\pi J^\mu$ and the fact that $T^{\mu\nu}_{(f)}$ is a quadratic function of $F^{\mu\nu}$, one takes the 4-divergence $T^{\mu\nu}_{,\nu}$ and finds that energy-momentum gained by fields is linear in the current J^μ and in the fields $F^{\mu\nu}$. This result does not rely on the specific form of (1), but just on the linearity of the Maxwell equations and on the quadratic dependence of $T^{\mu\nu}_{(f)}$ on fields.

The overall electromagnetic field can be split as follows:

$$F^{\mu\nu} = F^{\mu\nu}_{\text{ext}} + F^{\mu\nu}_{\text{int}} \tag{10}$$

where the first term is associated with external charges and the second one is related to the charge q whose equation of motion is discussed. In the case of continuously charged matter, interactions of an infinitesimal charge element belonging to q with other charged elements belonging to it are taken into account and every charged element interact with the entire field (10). As pointed out above, the elementary nature of a classical point charge rules out the calculation of such an interaction between its charged elements.

Therefore, in the classical electrodynamics of point charges one is left with the linear interaction of q with $F^{\mu\nu}_{\text{ext}}$ whereas an appropriate compensation has to be sought for the interaction with $F^{\mu\nu}_{\text{int}}$. The interaction with the retarded expression of this field depends on the specific motion of the charged particle. Therefore, it makes sense if one writes the appropriate substitute in terms of the particle's kinematic variables. This approach leads to the third-order LD equation (5) [see, e.g., Landau and Lifshitz, 1975, pp. 204–211, equations (76.1)–(76.2)].

4. A DILEMMA OF SECOND-ORDER EQUATIONS

Contrary to the structure of the LD equation (5), other authors have attempted to restore the second-order Newtonian form and suggested equations (6)–(9). In these equations the substitute for the contribution of $F^{\mu\nu}_{\text{int}}$ is written not in terms of the charge's kinematic variables alone, but in terms of these variables *and* of $F^{\mu\nu}_{\text{ext}}$ as well. This point becomes evident if one rewrites (6)–(9) as explicit expressions of a^{μ}.

This prescription raises the following question: a charge exchanges energy-momentum with its environment. As shown in (3) and derived also from the units used for the energy-momentum tensor, this quantity is *linear* in the particle's charge q and in the field $F^{\mu\nu}$. Therefore, it is not clear how an energy-momentum flux, which is *linear* in the fields, can be balanced by the matter equations, which are *nonlinear* in $F^{\mu\nu}_{\text{ext}}$. This dilemma is not settled by the authors who suggested (6)–(9). As a matter of fact, the counterarguments of Huschilt and Baylis (1974) and of Comay (1987) prove that equations (7)–(9) cannot settle this difficulty because of their inconsistency with energy conservation. In the following section it is proved that Eliezer's equation (6) also does not satisfy energy conservation.

5. THE ELIEZER EQUATION

Assume a charge moving rectilinearly along the x axis from $x=-\infty$ toward $x=\infty$ in an external static field. Along the x axis the external electric field takes the form $\mathbf{E}=(E_x, 0, 0)$. Let us evaluate the energy exchanged in the process. The static property of the external field proves that the self-energy of the external source does not vary during the process. Referring to the moving charge, let us integrate the 0-compound of Eliezer's equation (6) with respect to the invariant time τ. This is done separately for each term of the equation.

The integration of the first term yields

$$m\int_{-\infty}^{\infty} a^0\, d\tau = m\gamma(\infty) - m\gamma(-\infty) \tag{11}$$

where the 4-velocity $v^{\mu}=(\gamma, \gamma v, 0, 0)$, $\gamma=(1-v^2)^{-1/2}$, and v denotes the 3-velocity of the particle. This quantity represents the change of the self-energy of the moving charge during the entire process.

Integrating the next term, one finds

$$\begin{aligned} q\int_{-\infty}^{\infty} F^{0\nu}_{\text{ext}}\frac{dx_\nu}{d\tau}\, d\tau &= q\int_{-\infty}^{\infty} E_x\, dx \\ &= -q\Phi(\infty)+q\Phi(-\infty) \\ &= 0 \end{aligned} \tag{12}$$

Here the external static field E_x is derived from the potential Φ: $E_x = -\partial\Phi/\partial x$. The final null result is obtained from $\Phi(\pm\infty)=0$.

The next term yields straightforwardly

$$\frac{2q^3}{3m}\int_{-\infty}^{\infty}\frac{d(F_{\text{ext}}^{0\nu}v_\nu)}{d\tau}\,d\tau=\frac{2q^3}{3m}\,[F_{\text{ext}}^{0\nu}v_\nu(\infty)-F_{\text{ext}}^{0\nu}v_\nu(-\infty)]=0 \tag{13}$$

Here, as before, the vanishing outcome is obtained from the field's behavior at infinity.

The integration of the last term yields a quantity denoted by W_E,

$$W_E=\frac{2q^3}{3m}\int_{-\infty}^{\infty}(F_{\text{ext}}^{\alpha\beta}a_\alpha v_\beta)v^0\,d\tau \tag{14}$$

where, as mentioned above, $v^0=\gamma$.

These results should be compared with the energy radiated by the system. The general formula of energy radiated by a system where just one charge accelerates is [see Landau and Lifshitz (1975), p. 194, or Rohrlich (1965), p. 111, but notice the different metric used in the latter book]

$$W_R=-\frac{2q^2}{3}\int_{-\infty}^{\infty}(a^\alpha a_\alpha)v^0\,d\tau \tag{15}$$

An examination of (11)–(14) reveals that Eliezer's equation (6) is compatible with energy conservation if the integral (14) equals that of (15) multiplied by −1. Evidently, in a check of self-consistency of Eliezer's equation, one has to use the acceleration as calculated by the equation. This is accomplished by the replacement of a^α of (15) with Eliezer's acceleration (6).

Writing

$$\frac{d(F_{\text{ext}}^{\mu\nu}v_\nu)}{d\tau}=\frac{dF_{\text{ext}}^{\mu\nu}}{d\tau}\,v_\nu+F_{\text{ext}}^{\mu\nu}a_\nu \tag{16}$$

and substituting (6) into (15), one finds

$$a^\alpha a_\alpha=\frac{q}{m}\,F_{\text{ext}}^{\alpha\beta}v_\beta a_\alpha+\frac{2q^3}{3m^2}\,\frac{dF_{\text{ext}}^{\alpha\beta}}{d\tau}\,v_\beta a_\alpha+\frac{2q^3}{3m^2}\,F_{\text{ext}}^{\alpha\beta}a_\alpha a_\beta$$
$$+\frac{2q^3}{3m^2}\,(F_{\text{ext}}^{\gamma\delta}a_\gamma v_\delta)v^\alpha a_\alpha \tag{17}$$

An examination of the four terms on the right-hand side of (17) proves that the first term corresponds to W_E of (14). The third term vanishes identically because it is a contraction of an antisymmetric tensor $F_{\text{ext}}^{\alpha\beta}$ with a symmetric one $a_\alpha a_\beta$. Similarly, the last term vanishes because

$$2v^\alpha a_\alpha=\frac{d}{d\tau}\,v^\alpha v_\alpha=\frac{d}{d\tau}\,1=0$$

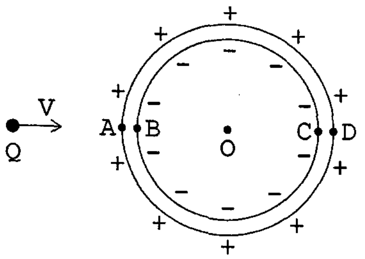

Fig. 1. A charge Q moves in the field of charges distributed uniformly on two concentric spherical shells.

The foregoing discussion proves that Eliezer's equation (6) conserves energy only if the following relation holds:

$$S = \int_{-\infty}^{\infty} \left(\frac{dF_{\text{ext}}^{\alpha\beta}}{d\tau} v_\beta a_\alpha \right) v^0 \, d\tau = 0 \tag{18}$$

On the other hand, one can refute (6) by means of an example where (18) is not satisfied. This is done in the following experiment.

Given a charge Q which moves along the x axis from $-\infty$ toward ∞. Two very large spherical shells are concentric at the origin. The spherical shells are made of insulating material and are covered uniformly with the same amount of positive and negative charges, respectively (see Figure 1). The difference between the radii of the shells is very small and the strength of the field between them is practically uniform. The electric field of the spherical shells vanishes at all space except at the region between the two shells. A description of this field as a function of points on the x axis can be seen on Figure 2. It shows four very short intervals $(A_1, A_2), \ldots, (D_1, D_2)$ around A, B. C, and D, respectively, where the field varies linearly between zero and the full strength prevailing between the two shells. The motion of the charge Q in the external field of the two spherical shells is considered.

Assume that the moving charge obeys Eliezer's equation (6). The evaluation of (18) is split into four integrals calculated along the very short intervals $(A_1, A_2), \ldots, (D_1, D_2)$. The sum of these four quantities is the

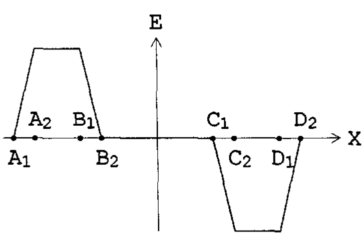

Fig. 2. The external field E_x drawn as a function of x. Scale of the x axis is not uniform and portions where the field varies are elongated.

required integral because at all other parts of the x axis, $dE_x/dx = 0$. Let us evaluate this integral at (A_1, A_2). The velocity of the moving charge does not vanish. As a result, due to the shortness of (A_1, A_2), the time duration taken by the moving charge to travel along this interval is also very short. It follows that, at (A_1, A_2), the contribution of the acceleration to the variation of velocity is very small and this velocity can be considered a constant there. Therefore, $v^0 = \gamma$ is a constant, too, and it can be taken out of the integral. Denoting the required integral by S_A, one finds

$$S_A = \int_{\tau(A_1)}^{\tau(A_2)} \left(\frac{dF_{\text{ext}}^{\alpha\beta}}{d\tau} v_\beta a_\alpha \right) v^0 \, d\tau = v^0 \int_{\tau(A_1)}^{\tau(A_2)} \left(\frac{dF_{\text{ext}}^{\alpha\beta}}{d\tau} v_\beta a_\alpha \right) d\tau \tag{19}$$

The integrand on the final form of (19) is an invariant. Therefore, it can be evaluated in the charge's rest frame, where one finds

$$v^\alpha = (1, 0, 0, 0) \tag{20}$$

$$a^\alpha = (0, a, 0, 0) \tag{21}$$

where a is the 3-acceleration in the rest frame. The charge moves along the x axis and a transformation into its rest frame does not alter E_x. Hence,

$$\frac{dF^{\mu\nu}}{d\tau} = \begin{pmatrix} 0 & -G & 0 & 0 \\ G & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{22}$$

where $G = dE_x/d\tau$. Substituting these results into (19), one obtains

$$S_A = -v^0 \int_{\tau(A_1)}^{\tau(A_2)} \frac{dE_x}{d\tau} a \, d\tau = -v^0 \frac{dE_x}{dx} \int_{A_1}^{A_2} a \, dx = -v^0 \frac{dE_x}{dx} I \tag{23}$$

where

$$I = \int_{A_1}^{A_2} a \, dx$$

and the integration is performed in the laboratory frame. At the interval (A_1, A_2), dE_x/dx is a constant, justifying its transfer out of the integration operation. Analogous expressions are obtained for the integrals S_B, S_C, and S_D, carried out on $(B_1, B_2), \ldots, (D_1, D_2)$, respectively.

An examination of (23) shows that the four integrals S_A, S_B, S_C, and S_D differ by a sign and by the factor $v^0 = \gamma$. This conclusion is derived from the fact that the absolute value $|dE_x/dx|$ is the same in the four cases and from the assumption that this relation holds also for $|I|$. Notice also that the charge's velocity at B equals that of C. Therefore, $S_B + S_C = 0$ because at B the acceleration and the field's derivative take opposite signs, whereas they have the same sign at C. It follows that in the present experiment the quantity defined in (18) is

$$S=[v^0(A)-v^0(D)]\left|\frac{dE_x}{dx}\right|I \tag{24}$$

Now assume that Eliezer's equation conserves energy. Hence, the charge's velocity at the end of the process must be smaller that its initial velocity. The corresponding difference in kinetic energy should balance the energy radiated. Hence, $v^0(A)>v^0(D)$. Therefore, an examination of (24) proves that it does not satisfy (18) and the assumption that Eliezer's equation (6) conserves energy leads to a contradiction.

The last result is obtained from the assumption that

$$|I|=\left|\int_{A_1}^{A_2} a\,dx\right| \tag{25}$$

takes the same value at the four intervals $(A_1, A_2), \ldots, (D_1, D_2)$. This is true if the acceleration a in the rest frame is the same at corresponding points of these intervals. Evidently, this assumption is accurate if the acceleration is obtained from the ordinary Lorentz force, because E_x does not vary under a Lorentz transformation associated with a velocity that is parallel to the x axis. Hence, the above-mentioned assumption concerning the integral I is inaccurate if Eliezer's equation is used instead of the ordinary Lorentz force. However, the difference $v^0(A)-v^0(D)$ of (24) is due to the radiation emitted during the entire process. In particular, this radiation takes place at the intervals (A_2, B_1) and (C_2, D_1) whose length is an independent parameter. Therefore, a correction due to Eliezer's radiation reaction term, which is introduced into the calculation of I, cannot compensate the discrepancy of (24). This argument completes the proof of the unphysical nature of Elizer's radiation reaction equation (6).

6. CONCLUSIONS

It is shown above that second-order equations of motion of charges that depend nonlinearly on external fields face a serious problem of energy conservation. The necessity of defining an energy-momentum tensor $T^{\mu\nu}$ in terms of the fields leads to a quadratic expression $T^{\mu\nu}(F^{\alpha\beta})$. Using Maxwell's equations, one finds that the 4-divergence of this tensor is *linear* in the current of charged matter and in the *overall* electromagnetic fields. These conclusions show that it is very unlikely that one can contrive an energy-momentum-conserving equation of motion of charges which is *nonlinear* in the *external* fields. Indeed, the work of Huschilt and Baylis (1974) and of Comay (1987) proves that equations (7)-(9) do not satisfy energy conservation. The present work discusses Eliezer's equation (6) and proves that, like other equations of its kind, it does not conserve energy. These conclusions provide an indirect support to the third-order LD equation (5).

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